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1984 J. Phys. A: Math. Gen. 17 2945

(http://iopscience.iop.org/0305-4470/17/15/012)

J. Phys. A: Math. Gen. 17 (1984) 2945–2954. Printed in Great Britain

The quantum mechanics of the supersymmetric nonlinear σ-model

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Received 29 May 1984

Abstract. The classical and quantum mechanical formalisms of the models are developed. The quantisation is done in such a way that the quantum theory can be represented explicitly in as simple a form as possible, and the problem of ordering of operators is resolved so as to maintain the supersymmetry algebra of the classical theory.

1. Introduction

In this paper we study the $N=1$ supersymmetric nonlinear σ-model in one (time) dimension on a Riemannian manifold M of metric g_{ij}. Setting out from a detailed analysis of the canonical formalism of the classical theory, we show how to formulate the quantum mechanical version of the model, performing the quantisation in such a way that as simple as possible a representation of the theory is obtained, and handling the order of operators problem so that the supersymmetry algebra is properly maintained.

The subject of supersymmetric quantum mechanics has seen continued interest starting with early work of Witten (1981), and of Salomonson and van Holten (1982) in which, amongst other things, the role of instantons in supersymmetry breaking is discussed. We note also more recent work by Abbott and Zakrzewski (1984) on this topic. There has been a review by Cooper and Freedman (1983) in which, in addition, the relationship of supersymmetric quantum mechanics to stochastic quantisation is developed. Important work of Witten (1982) introducing the index now generally called the Witten index, indicates how calculations done for supersymmetric quantum mechanics have implications for the question of supersymmetry breaking for the corresponding higher-dimensional field theories. The Witten index is discussed further in a paper by Lancaster (1982) in which quantum mechanical theories with extended supersymmetries are considered. In this context, we would cite the interesting paper of de Crumbrugghe and Rittenberg (1983) in which the possibilities for extended supersymmetries in quantum mechanics were surveyed. The present authors (Davis *et al* 1983a, 1984) have been interested in supersymmetric quantum mechanical models in which there are internal symmetries of either compact or non-compact nature. The interest in non-compact theories stems from the existence of hidden σ-model structures of non-compact nature within extended supergravity theories in four dimensions. In order to gain insight into the implications of the non-compactness of the symmetry, we have studied (Davis *et al* 1983b, 1984) analogous models in lower, i.e. two or even one, dimension.

The most general class of theories studied previously by the authors concerned $SO(N+1)$ or $SO(N, 1)$ σ-models, respectively of compact and non-compact type. The establishment of the quantum mechanical versions of these theories was vastly simplified by the use of coordinate systems in which the metric g_{ij} of the theory is proportional to the flat space metric T_{ij}, with $T_{ij}=\delta_{ij}$ for $SO(N+1)$, and such that $T_{00}=-1$ and $T_{ij}=\delta_{ij}$, $1\leq i, j\leq N$, for $SO(N, 1)$. In the present paper, we address the general problem of a σ-model on a manifold with a general metric g_{ij}. In a companion study, analogous results are presented for σ-models with extended $N=2$ supersymmetry.

As a preliminary, we study the classical version of the model, which through its Grassmann variable sector gives rise to constraints. We study the canonical formalism in terms of fundamental variables and their Dirac bracket structure. We use this formalism to discuss supersymmetry transformations and to derive the usual supersymmetry algebra. Further we identify a set of variables

$$x^i, \qquad R_j, \qquad \lambda^a_\alpha, \tag{1.1}$$

defined below, in terms of which the ordinary and Grassmann sectors of the set of canonical equations of the model decouple.

One begins the passage to quantum mechanics by replacing the real fundamental classical variables of the theory by the corresponding Hermitian quantum operators, and by replacing Dirac brackets of the former by the appropriate quantum brackets of the latter. This is a straightforward matter. Further, for the set (1.1), the only non-trivial results are

$$[x^i, R_j]=\mathrm{i}\delta^i_j, \tag{1.2}$$

$$\{\lambda^a_\alpha, \lambda^b_\beta\}=\eta^{ab}\delta_{\alpha\beta}, \tag{1.3}$$

where η^{ab} is the local flat space metric for M. Since there is a decoupling of the bosonic and fermionic variables, this allows a natural representation of R_j in terms of $-\mathrm{i}\,\partial/\partial x^j$ to be used along with a natural Fock space representation of the fermionic variables. As $1\leq\alpha, \beta\leq 2$, the latter is achieved by defining the annihilation operators

$$A^a=(\lambda^a_1+\mathrm{i}\lambda^a_2)/\sqrt{2}, \tag{1.4}$$

which obey

$$\{A^a, A^{b+}\}=\eta^{ab}. \tag{1.5}$$

This representation allows, in principle, the transition from the operator equations of the theory to an explicit (matrix) Schrödinger equation or generalised Laplace Beltami equation.

For the observables of the theory like Q_α, the supercharges, and H, the Hamiltonian, there is an operator ordering problem to be solved before suitable quantum expressions for these operators can be displayed. In fact, supersymmetry simplifies the problem. The demands that Q_α be Hermitian and reduce to the known classical formula in terms of fundamental variables in the classical limit fixes Q_α uniquely. There then remains the non-trivial task of showing that this definition satisfies the quantum mechanical supersymmetry algebra

$$Q_\alpha Q_\beta + Q_\beta Q_\alpha = 2\delta_{\alpha\beta} H_{\mathrm{qu}}, \tag{1.6}$$

and thereby identifying the (automatically Hermitian) expression for H_{qu}. Of course, H_{qu} so found should and does reduce to the known classical expression for H in terms of fundamental variables in the classical limit.

The content of the paper is summarised as follows. Section 2 reviews the classical description of the $N = 1$ supersymmetric model on a Riemannian manifold with metric g_{ij}. This discussion owes much to corresponding work in space–time of two dimensions, in which context see Freedman and Townsend (1981), Alvarez-Gaume and Freedman (1983) and Witten (1982). Section 3 develops the canonical formalism and discusses the supersymmetry algebra using it. Section 4 exhibits the choice of coordinates in which the algebra of the real and the Grassmann coordinates decouples. Section 5 discusses quantisation using the latter special set of coordinates to give a natural and especially simple representation of the quantum mechanics. Section 6 discusses the problem of operator ordering that must be solved to attain quantum mechanical formulae for the supercharges and the Hamiltonian that satisfy the usual supersymmetry algebra quantum mechanically. Operator ordering problems for (bosonic) σ-models have been studied before by Velo and Wess (1971) and by Charap (1973) but in many ways supersymmetry simplifies matters since solving the (simpler) problem for the supercharges turns out to be sufficient to provide a direct route to the required Hamiltonian.

2. The classical supersymmetric nonlinear σ-model

The bosonic sector of the model is defined in terms of n real fields $x^i(t)$, which are real functions from time onto a Riemannian manifold M with metric g_{ij}. The action is

$$S = \tfrac{1}{2}\int \mathrm{d}t\, g_{ij}\dot{x}^i\dot{x}^j \tag{2.1}$$

and this is invariant under coordinate reparametrisations of M.

To construct the $N = 1$ supersymmetric extension, we employ a pair of real Grassmann variables θ_α, $\alpha = 1, 2$, and introduce a set of n real superfields

$$\Phi^i = x^i + \mathrm{i}\theta_\alpha \varepsilon_{\alpha\beta}\psi^i_\beta + \tfrac{1}{2}\mathrm{i}\theta_\alpha\varepsilon_{\alpha\beta}\theta_\beta F^i. \tag{2.2}$$

Here $\varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha}$, $\varepsilon_{12} = 1$. Also, the ψ^i_β are real Grassmann variables. The standard supersymmetry transformations of superfields like the Φ^i are given by

$$\delta\Phi^i = -\varepsilon_\alpha \mathcal{Q}_\alpha \Phi^i, \tag{2.3}$$

where ε_α, $\alpha = 1, 2$, denote a pair of real Grassmann parameters, and $\mathcal{Q}_\alpha$ is given by

$$\mathcal{Q}_\alpha = \partial/\partial\theta_\alpha + \mathrm{i}\theta_\alpha \partial/\partial t. \tag{2.4}$$

In terms of components, (2.3) reads as

$$\delta x^i = -\mathrm{i}\varepsilon_\alpha\varepsilon_{\alpha\beta}\psi^i_\beta, \qquad \delta\psi^i_\alpha = -\varepsilon_\alpha F^i - \varepsilon_{\alpha\beta}\varepsilon_\beta \dot{x}^i, \qquad \delta F^i = \mathrm{i}\varepsilon_\alpha(\partial/\partial t)\psi^i_\alpha. \tag{2.5}$$

Introducing next the supercovariant derivatives

$$D_\alpha = \partial/\partial\theta_\alpha - \mathrm{i}\theta_\alpha\, \partial/\partial t, \tag{2.6}$$

which anticommute with the $\mathcal{Q}_\alpha$, we see that $D_\alpha \Phi^i$ possesses superfield transformation properties, as also does the superfield

$$\Delta = -\tfrac{1}{4} g_{ij}(\Phi) D_\alpha \Phi^i \mathrm{i}\varepsilon_{\alpha\beta} D_\beta \Phi^j. \tag{2.7}$$

From Δ, we construct an action

$$S = \int \mathrm{d}t\, L, \qquad L = \int \mathrm{d}^2\theta\, \Delta. \tag{2.8}$$

Since, from (2.8), we see that L is the 'F component' of Δ, the change δL in L under a supersymmetry transformation is given by

$$\delta L = \mathrm{i}(\partial/\partial t)\varepsilon_\alpha \Psi_\alpha, \tag{2.9}$$

where Ψ_α is the 'ψ component' of Δ

$$\Psi_\alpha = \tfrac{1}{4}\mathrm{i}\psi^k_\alpha g_{ij,k}\psi^i_\gamma \varepsilon_{\gamma\delta}\psi^j_\delta + \tfrac{1}{2} g_{ij} F^i \psi^j_\alpha - \tfrac{1}{2}\varepsilon_{\alpha\beta}\psi^i_\beta g_{ij}\dot{x}^j. \tag{2.10}$$

Hence S is a supersymmetric action, and is the required supersymmetric extension of (2.1). Calculation L as the 'F component' of Δ, we obtain the Lagrangian of the model in the form

$$\begin{aligned} L = {} & \tfrac{1}{2} g_{ij}(\dot{x}^i \dot{x}^j + F^i F^j + \mathrm{i}\psi^i_\alpha \dot{\psi}^j_\alpha) + g_{ij,kl}\psi^i_\alpha \varepsilon_{\alpha\beta}\psi^j_\beta \psi^k_\gamma \varepsilon_{\gamma\delta}\psi^l_\delta \\ & - \tfrac{1}{2}\mathrm{i}\psi^i_\alpha \varepsilon_{\alpha\beta}\psi^j_\beta \Gamma^p_{ij} g_{pk} F^k + \tfrac{1}{2}\mathrm{i} g_{ij}\psi^i_\alpha \Gamma^j_{pk}\dot{x}^p \psi^k_\alpha. \end{aligned} \tag{2.11}$$

In (2.11), the fields F^i are auxiliary fields which can be eliminated with the aid of their equations of motion

$$F^k = \tfrac{1}{2}\mathrm{i}\psi^i_\alpha \varepsilon_{\alpha\beta}\psi^j_\beta \Gamma^k_{ij}. \tag{2.12}$$

Hence, we have (Witten 1982, Freedman and Townsend 1981, Davis *et al* 1984)

$$L = \tfrac{1}{2} g_{ij}\dot{x}^i \dot{x}^j + \tfrac{1}{2}\mathrm{i} g_{ij}\psi^i_\alpha D\psi^j_\alpha + \tfrac{1}{12} R_{ip,jq}\psi^i_\alpha \varepsilon_{\alpha\beta}\psi^j_\beta \psi^p_\gamma \varepsilon_{\gamma\delta}\psi^q_\delta, \tag{2.13}$$

where

$$D\psi^j_\alpha = \dot{\psi}^j_\alpha + \Gamma^j_{pk}\dot{x}^p \psi^k_\alpha. \tag{2.14}$$

Further, we may calculate the supercharge Q_α corresponding to (2.5) via Noether's theorem written in the form

$$-\mathrm{i}\varepsilon_\alpha \varepsilon_{\alpha\beta} Q_\beta = \delta x^i\, \partial L/\partial \dot{x}^i + \delta\psi^i_\alpha\, \partial L/\partial \dot{\psi}^i_\alpha - \mathrm{i}\varepsilon_\alpha \Psi_\alpha. \tag{2.15}$$

Using (2.11) and (2.10), this leads to the result

$$Q_\alpha = \psi^i_\alpha g_{ij}\dot{x}^j. \tag{2.16}$$

3. Classical canonical formalism

From (2.13), we find the conjugate momenta

$$p_i = \partial L/\partial \dot{x}^i = g_{ij}\dot{x}^j + \tfrac{1}{2}\mathrm{i} g_{ij,k}\psi^j_\alpha \psi^k_\alpha, \tag{3.1}$$

$$\tau_i = \partial L/\partial \dot{\psi}^i = -\tfrac{1}{2}\mathrm{i} g_{ij}\psi^j. \tag{3.2}$$

It follows immediately from (3.2) that the canonical formalism involves constraint functions

$$\chi_i = \tau_i + \tfrac{1}{2}\mathrm{i}g_{ij}\psi^j. \tag{3.3}$$

This requires use of the formalism of Dirac, as generalised by Casalbuoni (1976) to take account of Grassmann variables. Using naive Poisson brackets

$$\{x^i, p_j\} = \delta^i_j, \qquad \{\psi^i, \tau_j\} = -\delta^i_j, \tag{3.4}$$

we obtain

$$\{\chi_i, \chi_j\} = c_{ij} = -\mathrm{i}g_{ij}, \tag{3.5}$$

and hence for any two field variables A and B define the Dirac bracket $\{A, B\}^*$. Explicitly we have

$$\{A, B\}^* = \{A, B\} - \{A, \chi_i\}\mathrm{i}g^{ij}\{\chi_j, B\}. \tag{3.6}$$

Within the Dirac bracket formalism, we may set $\chi_i = 0$ and calculate the basic Dirac brackets

$$\{x^i, p_j\}^* = \delta^i_j, \qquad \{\psi^i, \psi^j\}^* = -\mathrm{i}g^{ij}, \tag{3.7a, b}$$

$$\{\psi^i, p_j\}^* = -\tfrac{1}{2}g^{ik}g_{kl,j}\psi^l, \tag{3.7c}$$

$$\{p_i, p_j\}^* = -\tfrac{1}{4}\mathrm{i}g_{kn,i}g_{lm,j}g^{kl}\psi^n_\alpha\psi^m_\alpha, \tag{3.7d}$$

$$\{x^i, \psi_j\}^* = 0, \qquad \{x^i, x^j\}^* = 0. \tag{3.7e, f}$$

Other canonical calculations can be performed from (3.7) via

$$\{A, BC\}^* = \{A, B\}^*C \pm B\{A, C\}^*, \tag{3.8}$$

where the minus sign is required if and only if both of A and B are Grassmann. It is to be noted that the variables τ_i are no longer needed, as they can be eliminated from Dirac brackets, using $\chi_i = 0$. Also the Hamiltonian H corresponding to (2.13) is given by

$$H = \tfrac{1}{2}g^{ij}\pi_i\pi_j - \tfrac{1}{12}R_{ip,jq}\psi^i_\alpha\varepsilon_{\alpha\beta}\psi^j_\beta\psi^p_\gamma\varepsilon_{\gamma\delta}\psi^q_\delta, \tag{3.9}$$

where

$$\pi_i = p_i + \tfrac{1}{2}\mathrm{i}\Gamma^l_{ij}g_{lk}\psi^j_\alpha\psi^k_\alpha. \tag{3.10}$$

In what follows, any classical brackets employed will be Dirac brackets, so that the asterisk will hereafter be left implicit.

For many purposes, the results (3.7) are not the most convenient ones. Rather it is better to use π_i as given by (3.10) in the place of p_i. Hence it follows (3.7), that (3.7a, c and d) may be replaced by

$$\{x^i, \pi_j\} = \delta^i_j, \qquad \{\psi^i, \pi_j\} = -\Gamma^i_{jk}\psi^k, \tag{3.7g, h}$$

$$\{\pi_i, \pi_j\} = -\tfrac{1}{2}\mathrm{i}R_{ijpq}\psi^p_\alpha\psi^q_\alpha. \tag{3.7i}$$

Some manipulation is required to produce (3.7i). Also it follows from $\pi_i = g_{ij}\dot{x}^j$ that Q_α takes the simple form

$$Q_\alpha = \psi^i_\alpha\pi_i. \tag{3.11}$$

We turn first to the canonical view of the supersymmetry transformation (2.5). We may say that it is generated by Q_α if we have

$$\delta\Phi^i = \mathrm{i}\{\varepsilon_\alpha\varepsilon_{\alpha\beta}Q_\beta, \Phi^i\},$$

which requires

$$\{Q_\alpha, \Phi^i\} = -\mathrm{i}\varepsilon_{\alpha\beta}\mathscr{Q}_\beta\Phi^i. \tag{3.12}$$

The first two components of (3.12) are

$$\{Q_\alpha, x^i\} = -\psi^i_\alpha, \qquad \{Q_\alpha, \psi^i_\beta\} = -\mathrm{i}\delta_{\alpha\beta}\dot{x}^i - \mathrm{i}\varepsilon_{\alpha\beta}F^i. \tag{3.13a, b}$$

It is easy to check that insertion of (2.16) into the left sides of these equations and use of (3.7) leads to the right sides. The consistency of the same calculation for the 'F' component of (3.12) requires use of the equation of motion of ψ^i_α.

We turn next to the basic result of the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha\beta}H. \tag{3.14}$$

Calculation of the left side, using (2.16) and the canonical equations (3.7) does lead to the right side with H given by (3.9) upon use of a lemma. The lemma in question is (Alvarez-Gaume and Freedman 1983)

$$R_{ipjq}\psi^i_\alpha\varepsilon_{\alpha\beta}\psi^j_\beta\psi^p_\gamma\varepsilon_{\gamma\delta}\psi^q_\delta = -\tfrac{3}{2}R_{ijpq}\psi^i_\alpha\psi^j_\alpha\psi^p_\beta\psi^q_\beta, \tag{3.15}$$

which holds as a consequence of

$$\varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta} = \delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma} \tag{3.16}$$

and the cyclic property of the Riemann tensor.

4. Simplified canonical equations

The canonical equations

$$\{x^i, \pi_j\} = \delta^i_j, \qquad \{\psi^i, \psi^j\} = -\mathrm{i}g^{ij}, \qquad \{\psi^i, \pi_j\} = -\Gamma^i_{jk}\psi^k, \tag{3.7g, b, h}$$

$$\{\pi_i, \pi_j\} = -\tfrac{1}{2}\mathrm{i}R_{ij,pq}\psi^p_\alpha\psi^q_\alpha, \tag{3.7i}$$

$$\{x^i, \psi_j\} = 0, \qquad \{x^i, x^j\} = 0, \tag{3.7e, f}$$

are probably the most convenient set to use for many canonical calculations because Q_α and H as given by (3.11) and (3.9) are simple in terms of π_i and ψ^i_α. However, there is a simpler set of canonical equations, which separate the ordinary and Grassmann variables and which are crucial for passage to explicit representation of the quantum mechanics.

The first simplifying step is evident. We need the vielbein, defined by

$$e^i{}_a e^j{}_b g_{ij} = \eta_{ab}, \tag{4.1}$$

where η_{ab} is the local flat space metric corresponding to g_{ij}. Using η to raise and lower local indices we define the inverse vielbein by

$$e^i{}_a e_i{}^b = \delta^b_a. \tag{4.2}$$

Then it is natural to define

$$\lambda^a_\alpha = e_i{}^a \psi^i_\alpha, \tag{4.3}$$

so that

$$\psi^i_\alpha = e^i{}_a \lambda^a_\alpha. \tag{4.4}$$

Hence (3.7b) and (3.7e) become

$$\{\lambda^a_\alpha, \lambda^b_\beta\} = -\mathrm{i}\eta^{ab}\delta_{\alpha\beta}, \qquad \{x^i, \lambda^a_\alpha\} = 0. \tag{4.5b, e}$$

Further (3.7h) implies

$$\{\lambda^a_\alpha, \pi_i\} = \omega^a{}_{ib}\lambda^b_\alpha, \tag{4.6}$$

where $\omega_{aib} = -\omega_{bia}$ is the spin connection of the metric. By virtue of the metric postulate

$$D_i e_j{}^a = \partial_i e_j{}^a + \omega_{bi}{}^a e_j{}^b - \Gamma^k_{ij} e_k{}^a = 0,$$

it can be seen that the spin connection is given by

$$\omega_{aib} = e_{jb}(\partial_i e^j{}_a + \Gamma^j_{ik} e^k{}_a). \tag{4.7}$$

From (4.6), we are motivated to replace π_i by R_i given by

$$R_i = \pi_i - \tfrac{1}{2}\mathrm{i}\omega_{aib}\lambda^a_\beta\lambda^b_\beta, \tag{4.8}$$

so that we have

$$\{\lambda^a_\alpha, R_i\} = 0. \tag{4.5c}$$

The replacement of π_i by R_i also gives

$$\{x^i, R_j\} = \delta^i_j. \tag{4.5a}$$

It will be exactly what we seek if it gives rise to

$$\{R_i, R_j\} = 0 \tag{4.5d}$$

for then the set x^i, R_j, λ^a_α will satisfy the simplest possible canonical equations (4.5), in which the obvious

$$\{x^i, x^j\} = 0 \tag{4.5f}$$

is adjoined to those already discussed. A non-trivial calculation does serve to confirm that (4.5d) is satisfied.

5. Quantisation

The basic step in the creation of the quantum theory involves the replacement of real classical variables by Hermitian operators in quantum mechanics, and of the Dirac brackets of the classical theory by the appropriate quantum brackets divided by i. For the theory written in terms of

$$x^i, \qquad \pi_j, \qquad \psi^k_\alpha, \tag{5.1}$$

the non-trivial brackets are

$$[x^i, \pi_j] = \mathrm{i}\delta^i_j, \qquad \{\psi^i_\alpha, \psi^j_\beta\} = g^{ij}\delta_{\alpha\beta}, \tag{5.2a, b}$$

$$[\psi^i_\alpha, \pi_j] = -\mathrm{i}\Gamma^i_{jk}\psi^k_\alpha, \qquad [\pi_i, \pi_j] = \tfrac{1}{2}R_{ijpq}\psi^p_\alpha\psi^q_\alpha. \tag{5.2c, d}$$

No ambiguity should stem from the fact that, as usual, braces in a quantum theory denote anticommutators. For the theory written in terms of

$$x^i, \qquad R_j, \qquad \lambda^a_\alpha, \tag{5.3}$$

the non-trivial results are

$$[x^i, R_j] = \mathrm{i}\delta^i_j, \qquad \{\lambda^a_\alpha, \lambda^b_\beta\} = \eta^{ab}\delta_{\alpha\beta}. \tag{5.4a, b}$$

It should be stressed that there are no ambiguities of orderings of operators on the right sides of (5.3) and (5.4) although for (5.2d) this is true only because the Riemann tensor is antisymmetric in its last two indices.

The set (5.3) of variables is important because the quantum mechanical representation of it is immediate. Firstly we have for R_i the result

$$R_i = g^{-1/4}(-\mathrm{i}\,\partial/\partial x^i)g^{1/4} \tag{5.5}$$

where, as is well known, the factors involving $g = \det g$ are required by Hermiticity. Secondly setting

$$A^a = (\lambda^a_1 + \mathrm{i}\lambda^a_2)/\sqrt{2}, \tag{5.6}$$

we get from (5.4b)

$$\{A^a, A^{b\dagger}\} = \eta^{ab}. \tag{5.7}$$

It follows that the A^a are a set of n fermionic annihilation operators. Since they commute with x^i and R^j, an evident fermionic Fock space representation of them therefore exists. An explicit representation of π_i now follows (4.8) and one of ψ^i_α follows (4.4). The representation (4.8) affords of π_i is well defined quantum mechanically in virtue of the antisymmetry property of the spin connection.

The second stage in the setting up of the quantum mechanical version of the $N = 1$ supersymmetric nonlinear σ-models consists of giving appropriate generalisations of the classical expression (2.16) or (3.11) for Q_α and of (3.9) of H. Appropriate generalisations are furnished by operators which

(a) are Hermitian,

(b) satisfy the supersymmetry algebra (3.14), and

(c) reduce to the corresponding classical results

when questions of non-commutativity of operators are ignored. They are the subject of the next section.

6. Quantum mechanical definition of Q_α and H

It is clear that the real classical supercharge Q_α of (3.11) must be replaced in the quantum theory by the Hermitian operator

$$Q_\alpha = \tfrac{1}{2}(\psi^i_\alpha\pi_i + \pi_i\psi^i_\alpha). \tag{6.1}$$

To justify this, it is necessary to show that

$$\{Q_\alpha, Q_\beta\} = Q_\alpha Q_\beta + Q_\beta Q_\alpha = 2\delta_{\alpha\beta}H_{\mathrm{qu}} \tag{6.2}$$

follows (5.2) and thereby identify H_{qu}. In fact it is simpler first to calculate H_{qu} using

$$H_{qu}=\tfrac{1}{4}\{Q_\alpha, Q_\alpha\} \tag{6.3}$$

and to establish the stronger result (6.2) thereafter. In the calculation of H_{qu} using (6.3) terms of three types arise.

(A) From use of $\{\psi^i_\alpha, \psi^j_\beta\}$, we get

$$\tfrac{1}{8}g^{ij}\pi_i\pi_j+\tfrac{1}{4}\pi_i g^{ij}\pi_j+\tfrac{1}{8}\pi_i\pi_j g^{ij}.$$

Direct calculation based on (5.2a) equates this to

$$\tfrac{1}{2}g^{-1/4}\pi_i g^{ij}g^{1/2}\pi_j g^{-1/4}-\tfrac{1}{8}R+\tfrac{1}{16}g^{ij}{}_{,k}\Gamma^k_{ij}, \tag{6.4}$$

where R is the curvature scalar. Passage to a result of this latter sort follows the usual discussion of the quantum mechanics of a bosonic particle on a Riemannian manifold (de Witt 1957, Charap 1973, Schulman 1981).

(B) From use of $[\pi_i, \psi^j_\beta]$ and $[\psi^i_\alpha, \pi_j]$, we get quite easily a total contribution which cancels the last term of (6.4).

(C) From use of $[\pi_i, \pi_j]$, we get

$$\tfrac{1}{8}R+\tfrac{1}{8}R_{ijpq}\psi^i_\alpha\psi^j_\alpha\psi^p_\beta\psi^q_\beta. \tag{6.5}$$

Hence, we get

$$H_{qu}=\tfrac{1}{2}g^{-1/4}\pi_i g^{1/2}g^{ij}\pi_j g^{-1/4}+\tfrac{1}{8}R_{ijpq}\psi^i_\alpha\psi^j_\alpha\psi^p_\beta\psi^q_\beta. \tag{6.6}$$

The results (6.6) for H_{qu} and (3.9) for the classical Hamiltonian may be compared. Passing from (6.6) to a classical limit agrees exactly with the insertion of the classical lemma (3.15) into (3.9). However, if we repeat the classical proof of (3.15) in the quantum theory, we find that an additional term $(-R)$ should be supplied on the right-hand side of the equality. No doubt (6.6) is the simplest form to use in the quantum mechanics.

Upon attempting to generalise the above calculation so as to establish (6.2), one finds $\{\psi^i_\alpha, \psi^j_\beta\}$ gives a $\delta_{\alpha\beta}$ structure directly under (A), and terms (B) do the same job as they did above. Terms (C) cause problems. They lead directly to

$$\tfrac{1}{8}R\delta_{\alpha\beta}+\tfrac{1}{8}R_{ijpq}\psi^i_\alpha\psi^j_\beta\psi^p_\gamma\psi^q_\gamma.$$

To show that the last term is proportional to $\delta_{\alpha\beta}$, in which case the same H_{qu} emerges this way as before, we contract this term in turn with $\sigma^k_{\alpha\beta}$ for $k=1,2,3$. For $k=2$, σ^2 is antisymmetric and gives zero directly. For $k=1$, we employ a consequence of the completeness of the Pauli matrices

$$\sigma^1_{\alpha\beta}\delta_{\gamma\delta}+\sigma^1_{\gamma\delta}\delta_{\alpha\beta}=\sigma^1_{\alpha\delta}\delta_{\beta\gamma}+\sigma^1_{\beta\gamma}\delta_{\alpha\delta},$$

along with (5.2b) and standard properties of the Riemann tensor to get the required vanishing result. The same proof applies to $k=3$, and our proof of (6.2) is finished.

Having justified the quantum mechanical expression (6.1) for the supercharge Q_α, we may use (6.1) and the commutation relations (5.1) to calculate $[Q_\alpha, x^i]$ and $\{Q_\alpha, \psi^i_\beta\}$ in the quantum mechanics. The results are seen to be compatible with (3.13) provided that one makes the identification of $\dot{x}^i$ with the Hermitian operator $\tfrac{1}{2}(g^{ij}\pi_i+\pi_i g^{ij})$.

Applications of the results of the present paper, e.g. to the solution of the Schrödinger equation of the supersymmetric nonlinear σ-model on a Riemannian manifold or to the solution of $Q_\alpha\Psi=0$ for the zero energy eigenstates of H_{qu}, will be returned to elsewhere. For the latter context, we note the following. Equations (6.1) and (5.2*c*) give

$$Q_\alpha = \psi^i_\alpha \pi_i + \tfrac{1}{2}\mathrm{i}\psi_\alpha{}^k \Gamma^l_{kl}.$$

Then using (5.5), (4.3) and (4.8), we derive the result

$$Q_\alpha = -\mathrm{i}\lambda^a_\alpha D_a, \tag{6.7}$$

where

$$D_a = e^i{}_a(\partial/\partial x^i - \tfrac{1}{2}\omega_{aib}[\lambda^a, \lambda^b]). \tag{6.8}$$

Acknowledgment

We thank St John's College, Cambridge, for support of a visit of J W van Holten to Cambridge during which work on the present paper was done. A C Davis and P Popat are indebted to the SERC (UK), the former for an Advanced Fellowship, the latter for a Research Studentship.

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